

JET SCHEMES AND INVARIANT THEORY

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ABSTRACT. Given a complex, reductive algebraic group G and a G -module V , the m th jet scheme G_m acts on the m th jet scheme V_m for all $m \geq 1$. There is a natural map $\mathcal{O}((V//G)_m) \rightarrow \mathcal{O}(V_m)^{G_m}$ which in general fails to be an isomorphism. Under some conditions a stabilization phenomenon occurs at the level of arc spaces, and the induced map $\mathcal{O}((V//G)_\infty) \rightarrow \mathcal{O}(V_\infty)^{G_\infty}$ is an isomorphism. We give a sufficient criterion for this to occur, and prove that it holds when $G = \mathrm{SL}_n, \mathrm{GL}_n, \mathrm{SO}_n$, or Sp_{2n} and V is a sum of copies of the standard representation and its dual.

1. INTRODUCTION

There has been some recent interest in jet schemes and arc spaces due to a seminal paper by Nash [N] in 1996. Given an irreducible scheme X of finite type over an algebraically closed field k , the m th jet scheme X_m is determined by its functor of points. For every k -algebra A , we have a bijection

$$\mathrm{Hom}(\mathrm{Spec}(A), X_m) \cong \mathrm{Hom}(\mathrm{Spec}(A[t]/\langle t^{m+1} \rangle), X).$$

For a smooth, irreducible scheme X , X_m is irreducible for all $m \geq 1$ and is a vector bundle of rank $m \dim(X)$ over X . If X is singular, the jet schemes are much more subtle and carry information about the singularities of X . For an irreducible X , an important problem is to determine when X_m is irreducible for all m . By a result of Mustata [Mu], this holds when X is locally a complete intersection with rational singularities. On the other hand, this is not a necessary condition as Example 4.7 of [Mu] shows.

If $p > m$, we have projections $\pi_{p,m} : X_p \rightarrow X_m$ which are compatible when defined. The arc space X_∞ is defined to be $\lim_{\leftarrow m} X_m$, and there are natural maps $\psi_m : X_\infty \rightarrow X_m$. In this paper we will always work over the field \mathbb{C} , so for every m , there exists $p \geq m$ such that $\psi_m(X_\infty) = \pi_{p,m}(X_p)$ (see Corollary 5.9 of [EM]). Even though it is generally not of finite type, X_∞ has some nicer properties than X_m . For example it is always irreducible by a theorem of Kolchin [Kol]. Moreover, there is a close connection between the theory of arc spaces and the theory of vertex algebras. An *abelian* vertex algebra is just a commutative algebra equipped with a derivation, and for any scheme X , the ring $\mathcal{O}(X_\infty)$ has a natural derivation D which gives it the structure of an abelian vertex algebra [FBZ]. There are various ways to recover the ring $\mathcal{O}(X)$; one way is to take one-point conformal blocks over \mathbb{P}^1 (see Section 8.4.4 of [FBZ]), and another way is to apply the Zhu functor from vertex algebras to associative algebras [Zh]. Given a vertex algebra \mathcal{V} with a good increasing filtration, the associated graded object $gr(\mathcal{V})$ is an abelian vertex algebra which can often be interpreted as $\mathcal{O}(X_\infty)$ for some X . In this case, the geometry of X_∞ carries important information about the structure of \mathcal{V} .

Our main goal is to establish some foundational results on the interaction between jet schemes, arc spaces, and classical invariant theory. If G is a complex reductive algebraic

group, G_m is an algebraic group which is a unipotent extension of G . If V is a finite-dimensional representation of G , there is an action of G_m on V_m , and we are interested in the invariant ring $\mathcal{O}(V_m)^{G_m}$. Let $V//G$ denote the affine variety corresponding to $\mathcal{O}(V)^G$. There is a natural map

$$(1.1) \quad \mathcal{O}((V//G)_m) \rightarrow \mathcal{O}(V_m)^{G_m}$$

which was studied in some special cases by Eck in [E] and by Frenkel-Eisenbud in the appendix of [Mu]. In these examples, (1.1) is an isomorphism for all $m \geq 1$, but this is not true in general. We give some necessary and some sufficient conditions for (1.1) to be an isomorphism for all $m \geq 1$, and we show that a new phenomenon can occur: the induced map

$$(1.2) \quad \mathcal{O}((V//G)_\infty) \rightarrow \mathcal{O}(V_\infty)^{G_\infty}$$

can be an isomorphism even if (1.1) fails to be an isomorphism for finite values of m . We show that this holds for $G = \mathrm{SL}_n, \mathrm{GL}_n, \mathrm{SO}_n$, or Sp_{2n} and V is a sum of copies of the standard representation and its dual. Since $\mathcal{O}((V//G)_\infty)$ is generated by $\mathcal{O}(V//G)$ as a differential algebra, $\mathcal{O}(V_\infty)^{G_\infty}$ is *finitely generated* as a differential algebra in these cases. An interesting problem is to find sufficient conditions for $\mathcal{O}(V_\infty)^{G_\infty}$ to be finitely generated as a differential algebra even if (1.2) is not an isomorphism.

Our results have an application to the *vertex algebra commutant problem*, which we develop in a separate paper [LSS]. Given a vertex algebra \mathcal{V} and a subalgebra $\mathcal{A} \subset \mathcal{V}$, $\mathrm{Com}(\mathcal{A}, \mathcal{V})$ is defined to be the set of elements $v \in \mathcal{V}$ such that $[a(z), v(w)] = 0$ for all $a \in \mathcal{A}$. This construction is analogous to the ordinary commutant in the theory of associative algebras, and is important in physics in the construction of coset conformal field theories. In our main examples, \mathcal{V} is the $\beta\gamma$ system, or algebra of chiral differential operators on a finite-dimensional G -module V , and \mathcal{A} is the affine vertex algebra associated to the Lie algebra \mathfrak{g} of G . The associated graded algebra $\mathrm{gr}(\mathcal{V})$ is isomorphic to $\mathcal{O}((V \oplus V^*)_\infty)$ as differential commutative algebra. When $G = \mathrm{SL}_n, \mathrm{GL}_n, \mathrm{SO}_n$ or Sp_{2n} and V is a sum of copies of the standard representation, we have

$$\mathrm{gr}(\mathrm{Com}(\mathcal{A}, \mathcal{V})) \cong \mathcal{O}((V \oplus V^*)_\infty)^{G_\infty} \cong \mathcal{O}(((V \oplus V^*)//G)_\infty).$$

It follows that in these examples, a minimal generating set for $\mathcal{O}(V \oplus V^*)^G$ gives rise to a minimal strong generating set for $\mathrm{Com}(\mathcal{A}, \mathcal{V})$ as a vertex algebra. Moreover, all normally ordered polynomial relations among the generators of $\mathrm{Com}(\mathcal{A}, \mathcal{V})$ and their derivatives are consequences of relations among the generators of $\mathcal{O}(V \oplus V^*)^G$ and their derivatives, with suitable quantum corrections.

2. JET SCHEMES

We recall some basic facts about jet schemes, following the notation in [EM]. Let X be an irreducible scheme of finite type over \mathbb{C} . For each integer $m \geq 0$, the jet scheme X_m is determined by its functor of points: for every \mathbb{C} -algebra A , we have a bijection

$$\mathrm{Hom}(\mathrm{Spec}(A), X_m) \cong \mathrm{Hom}(\mathrm{Spec}(A[t]/\langle t^{m+1} \rangle), X).$$

Thus the \mathbb{C} -valued points of X_m correspond to the $\mathbb{C}[t]/\langle t^{m+1} \rangle$ -valued points of X . If $p > m$, we have projections $\pi_{p,m} : X_p \rightarrow X_m$ which are compatible when defined: $\pi_{p,m} \circ \pi_{q,p} = \pi_{q,m}$. Clearly $X_0 = X$ and X_1 is the total tangent space $\mathrm{Spec}(\mathrm{Sym}(\Omega_{X/\mathbb{C}}))$. The assignment $X \mapsto X_m$ is functorial, and a morphism $f : X \rightarrow Y$ induces $f_m : X_m \rightarrow Y_m$ for

all $m \geq 1$. If X is nonsingular, X_m is irreducible and nonsingular for all m . Moreover, if X, Y are nonsingular and $f : Y \rightarrow X$ is a smooth surjection, f_m is surjective for all m .

If $X = \text{Spec}(R)$ where $R = \mathbb{C}[y_1, \dots, y_r]/\langle f_1, \dots, f_k \rangle$, we can find explicit equations for X_m . Define new variables $y_j^{(i)}$ for $i = 0, \dots, m$, and define a derivation D by $D(y_j^{(i)}) = y_j^{(i+1)}$ for $i < m$, and $D(y_j^{(m)}) = 0$, which specifies its action on all of $\mathbb{C}[y_1^{(0)}, \dots, y_r^{(m)}]$. In particular, $f_\ell^{(i)} = D^i(f_\ell)$ is a well-defined polynomial in $\mathbb{C}[y_1^{(0)}, \dots, y_r^{(m)}]$. Letting $R_m = \mathbb{C}[y_1^{(0)}, \dots, y_r^{(m)}]/\langle f_1^{(0)}, \dots, f_k^{(m)} \rangle$, we have $X_m \cong \text{Spec}(R_m)$. By identifying y_j with $y_j^{(0)}$, we see that R is naturally a subalgebra of R_m . There is a $\mathbb{Z}_{\geq 0}$ -grading $R_m = \bigoplus_{n \geq 0} R_m[n]$ by weight, defined by $\text{wt}(y_j^{(i)}) = i$. For all m , $R_m[0] = R$ and $R_m[n]$ is a module over R .

Given a scheme X , define $X_\infty = \lim_{\leftarrow m} X_m$, which is known as the *arc space* of X . For a \mathbb{C} -algebra A , we have a bijection $\text{Hom}(\text{Spec}(A), X_\infty) \cong \text{Hom}(\text{Spec} A[[t]], X)$. We denote by ψ_m the natural map $X_\infty \rightarrow X_m$. If $X = \text{Spec}(R)$ as above,

$$X_\infty \cong \text{Spec}(R_\infty), \text{ where } R_\infty = \mathbb{C}[y_1^{(0)}, \dots, y_j^{(i)}, \dots]/\langle f_1^{(0)}, \dots, f_\ell^{(i)}, \dots \rangle.$$

Here $i = 0, 1, 2, \dots$ and $D(y_j^{(i)}) = y_j^{(i+1)}$ for all i . By a theorem of Kolchin [Kol], X_∞ is irreducible whenever X is irreducible.

3. GROUP ACTIONS ON JET SCHEMES

We establish some elementary properties of jet schemes and quotient mappings for reductive group actions. Mainly we see what one can say using Luna's slice theorem [Lun].

Let G be a complex reductive algebraic group with Lie algebra \mathfrak{g} . For $m \geq 1$, G_m is an algebraic group which is the semidirect product of G with a unipotent group U_m . The Lie algebra of G_m is $\mathfrak{g}[t]/t^{m+1}$. Given an affine G -variety Y , there is the quotient $Z := Y//G = \text{Spec}(\mathcal{O}(Y)^G)$ and the canonical map $p : Y \rightarrow Z$ (sometimes denoted p_Y) which is dual to the inclusion $\mathcal{O}(Y)^G \subset \mathcal{O}(Y)$. We have a natural action of G_m on Y_m , and we are interested in the invariant ring $\mathcal{O}(Y_m)^{G_m}$, the morphism $p_m : Y_m \rightarrow Z_m$ and whether or not $p_m^* \mathcal{O}(Z_m) = \mathcal{O}(Y_m)^{G_m}$. If $\varphi : X \rightarrow Y$ is a morphism of affine G -varieties, then $\varphi//G$ will denote the induced mapping of $X//G$ to $Y//G$.

Recall that a morphism of varieties is étale if it is smooth with fibers of dimension zero. If $\varphi : X \rightarrow Y$ is a morphism where X and Y are smooth, then φ is étale if and only if $d\varphi_x : T_x X \rightarrow T_{\varphi(x)} Y$ is an isomorphism for all $x \in X$.

Definition 3.1. Let G be a reductive complex algebraic group and let $\varphi : X \rightarrow Y$ be an equivariant map of affine G -varieties. We say that φ is excellent if the following hold.

- (1) φ is étale.
- (2) $\varphi//G : X//G \rightarrow Y//G$ is étale.
- (3) The canonical map $(\varphi, p_X) : X \rightarrow Y \times_{Y//G} X//G$ is an isomorphism.

We will write $\mathcal{O}(X_m)^{G_m} = \mathcal{O}((X//G)_m)$ as shorthand for $\mathcal{O}(X_m)^{G_m} = p_m^* \mathcal{O}((X//G)_m)$. Let us say that X is m -bad if $\mathcal{O}(V_m)^{G_m} \neq \mathcal{O}((V//G)_m)$. Otherwise we say that X is m -good. Usually we drop the m . We say that X is D -finite if $\mathcal{O}(X_\infty)^{G_\infty}$ is finitely generated as a differential algebra.

Lemma 3.2. *Suppose that $\varphi: X \rightarrow Y$ is excellent and surjective. Then X is good (resp. D -finite) if and only if Y is good (resp. D -finite).*

Proof. Since φ is étale we have that $X_m \simeq X \times_Y Y_m$. Since φ is excellent we have that

$$X \times_Y Y_m \simeq X//G \times_{Y//G} Y \times_Y Y_m \simeq X//G \times_{Y//G} Y_m$$

and since $\varphi//G$ is étale we have that $(X//G)_m \simeq X//G \times_{Y//G} (Y//G)_m$. If $\mathcal{O}(Y_m)^{G_m} = \mathcal{O}((Y//G)_m)$, then

$$\mathcal{O}(X_m)^{G_m} \simeq \mathcal{O}(X//G) \otimes_{\mathcal{O}(Y//G)} \mathcal{O}(Y_m)^{G_m} \simeq \mathcal{O}(X//G) \otimes_{\mathcal{O}(Y//G)} \mathcal{O}((Y//G)_m) \simeq \mathcal{O}((X//G)_m).$$

Conversely, if φ is surjective and $\mathcal{O}(Y_m)^{G_m} \neq \mathcal{O}((Y//G)_m)$, then, since $\varphi//G$ is faithfully flat, we have that $\mathcal{O}(X//G) \otimes_{\mathcal{O}(Y//G)} \mathcal{O}(Y_m)^{G_m} \neq \mathcal{O}(X//G) \otimes_{\mathcal{O}(Y//G)} \mathcal{O}((Y//G)_m)$ and hence that $\mathcal{O}(X_m)^{G_m} \neq \mathcal{O}((X//G)_m)$. Hence X is good if and only if Y is good.

Now

$$\mathcal{O}(X_\infty)^{G_\infty} \cong \mathcal{O}(Y_\infty)^{G_\infty} \otimes_{\mathcal{O}(Y)^G} \mathcal{O}(X)^G.$$

Thus if Y is D -finite, then clearly so is X . Conversely, assume that X is D -finite. Set $A := \mathcal{O}(X_\infty)^{G_\infty}$. Then we have the weight grading $A = \bigoplus_{n \in \mathbb{N}} A_n$ where $A_0 = \mathcal{O}(X)^G$. Let B denote $\mathcal{O}(Y_\infty)^{G_\infty}$. Then B is graded and the isomorphism $A \cong B \otimes_{\mathcal{O}(Y)^G} \mathcal{O}(X)^G$ is an isomorphism of graded rings. Let $f_i \otimes h_i$ be generators of $A \simeq B \otimes_{\mathcal{O}(Y)^G} \mathcal{O}(X)^G$ as differential graded algebra. We may assume that each f_i has weight n_i for some $n_i \in \mathbb{N}$. Let p_1, \dots, p_d be generators of $\mathcal{O}(X)^G$. Then $Dp_j = \sum f_{ij} \otimes h_{ij}$ where the h_{ij} are elements of $\mathcal{O}(X)^G$ and the f_{ij} are in B_1 . Now take the collection of elements f_i and f_{ij} in B . Then an induction argument shows that D applied repeatedly to the elements $f_i \otimes h_i$ ends up in the $\mathcal{O}(X)^G$ -submodule of A generated by D applied to products of the elements f_i and f_{ij} . Since φ is faithfully flat, this shows that the B_0 -submodule of B_n generated by the elements f_i and f_{ij} is B_n since this submodule tensored with $\mathcal{O}(X)^G$ is A_n . Hence the f_i and f_{ij} generate B_n for all n and Y is D -finite. \square

A subset S of X is G -saturated if $S = p^{-1}(p(S))$, equivalently; S is a union of fibers of p .

Corollary 3.3. (1) *Suppose that $X = \bigcup X_\alpha$ where the X_α are open and G -saturated. Then X is good (resp. D -finite) if and only if each X_α is good (resp. D -finite).*

(2) *Let W be a G -module and $U = W_f$ where $f \in \mathcal{O}(W)^G$ and $f(0) \neq 0$. Then W is good (resp. D -finite) if and only if U is good (resp. D -finite).*

Proof. For (1) we may assume that we have a finite cover. Then $\coprod X_\alpha \rightarrow X$ is excellent and surjective and (1) follows from Lemma 3.2. For (2) we may assume that U is good (resp. D -finite). Now W is the union of U and finitely many translates U_λ where $U_\lambda = \lambda \cdot U$ for $\lambda \in \mathbb{C}^*$. Clearly each U_λ is good (resp. D -finite) since U is. Thus we can apply (1). \square

Lemma 3.4. *Let H be a reductive subgroup of G and Y an affine H -variety. Then*

$$\mathcal{O}((G \times^H Y)_m)^{G_m} \simeq \mathcal{O}(Y_m)^{H_m}.$$

Hence $G \times^H Y$ is good (resp. D -finite) if and only if Y is good (resp. D -finite).

Proof. We have that $G \times^H Y$ is the quotient of $G \times Y$ by the H -action sending (g, y) to (gh^{-1}, hy) for $(g, y) \in G \times Y$ and $h \in H$. We also have an action of G on the left on $G \times Y$ which commutes with the action of H and induces the G -action on $G \times^H Y$. Now $G \times Y \rightarrow G \times^H Y$ is a principal H -bundle. For a trivial principal H bundle $U \times H$, we

have that $(U \times H)_m = U_m \times H_m$ is a trivial H_m -bundle with quotient U_m . Thus we see that $(G \times^H Y)_m$ is the quotient of $G_m \times Y_m$ by the action of H_m (it is a principal bundle).

Consider the action of G_m on $G_m \times Y_m$. Then the quotient is clearly just projection to Y_m , so that $\mathcal{O}(G_m \times Y_m)^{G_m} \simeq \mathcal{O}(Y_m)$. Thus $\mathcal{O}(G_m \times Y_m)^{G_m \times H_m} \simeq \mathcal{O}(Y_m)^{H_m}$ so that $\mathcal{O}((G \times^H Y)_m)^{G_m} \simeq \mathcal{O}(Y_m)^{H_m}$. The lemma follows since $(G \times^H Y) // G \simeq Y // H$. \square

Let X be a smooth affine G -variety and suppose that Gx is a closed orbit. Then the isotropy group $H := G_x$ is reductive and we have a splitting of H -modules $T_x X = T_x(Gx) \oplus N$. The representation (N, H) is called *the slice representation at x* . Here is Luna's slice theorem [Lun] in our context.

Theorem 3.5. (1) *There is a locally closed affine H -stable and H -saturated subvariety S of X containing x such that $U := G \cdot S$ is a G -saturated affine open subset of X . Moreover, the canonical G -morphism*

$$\varphi: G \times^H S \rightarrow U, \quad [g, s] \mapsto gs$$

is excellent.

(2) *S is smooth at x and the H -modules $T_x S$ and N are isomorphic. Possibly shrinking S we can arrange that there is an excellent surjective H -morphism $\psi: S \rightarrow N_f$ which sends x to 0, inducing an excellent G -morphism*

$$\tau: G \times^H S \rightarrow G \times^H N_f$$

where $f \in \mathcal{O}(N)^H$ and $f(0) \neq 0$.

Combining 3.2–3.5 we obtain

Corollary 3.6. *Suppose that X is smooth. Let (W, H) be a slice representation of X .*

- (1) *If X is good (resp. D -finite) then so is W .*
- (2) *If W is good (resp. D -finite) for each slice representation (W, H) of X , then X is good (resp. D -finite).*

Lemma 3.7. *Suppose that $V \oplus W$ is a representation of G . If W is bad (resp. not D -finite) then so is $V \oplus W$.*

Proof. If W is bad there is a G_m -invariant polynomial on W_m which does not come from $\mathcal{O}((W // G)_m)$. Then clearly it cannot come from an element of $\mathcal{O}(((V \oplus W) // G)_m)$. Now minimal generators of $\mathcal{O}(V_\infty \oplus W_\infty)^{G_\infty}$ can clearly be chosen to be bihomogeneous in the variables of V_∞ and W_∞ . Thus if $V \oplus W$ is D -finite, then so is W . \square

The results above say that a representation is bad (resp. not D -finite) if a subrepresentation or slice representation is bad (resp. not D -finite). Now let us consider some examples.

Example 3.8. Let $(V, G) = (\mathbb{C}, \pm 1)$. Then V is bad. Let z be a coordinate function on V . Then V_1 has coordinates $z = z^{(0)}$ and $z^{(1)}$. The invariants of $G = G_1$ are generated by z^2 , $zz^{(1)}$ and $(z^{(1)})^2$. The invariants coming from the quotient are z^2 and $2zz^{(1)}$. If one goes to degree 2, then from $\mathbb{C}[z^2]$ we get z^2 , $2zz^{(1)}$ and $2(z^{(1)})^2 + 2zz^{(2)}$. But among the G_2 -invariants we have z^2 , $zz^{(1)}$, $zz^{(2)}$, $(z^{(1)})^2$, $z^{(1)}z^{(2)}$ and $(z^{(2)})^2$. Things are only getting worse. See Theorem 3.11 for the general case.

Example 3.9. Let $G = \mathbb{C}^*$ and let $V = \mathbb{C}^2$ with weights 2 and -3 . Then $\mathcal{O}(V)^G$ is generated by $z = x^3y^2$, so that $X = V//G \cong \mathbb{C}$. For $m = 1$, $w := (D(z))^2/z = x(3yx^{(1)} + 2xy^{(1)})^2$ is not a function on X_1 , but it is a G_1 -invariant function on V_1 . Hence V is 1-bad. In fact, it is m -bad for any $m \geq 1$. See Theorem 3.13 for a generalization. Computer calculations suggest that V is D -finite with generators z and w . Thus this is likely an example where V is bad yet D -finite.

Example 3.10. Let $G = \mathrm{SO}_3$ and let V be the direct sum of 6 copies of the standard representation \mathbb{C}^3 , with basis $\{x_j, y_j, z_j \mid j = 1, \dots, 6\}$. The generators of $\mathcal{O}(V)^G$ are quadratics q_{ij} corresponding to a choice of two indices i, j which need not be distinct, and cubics c_{klm} corresponding to a choice of three distinct indices k, l, m . Define

$$f = \sum_{\sigma \in \mathfrak{S}_6} \mathrm{sgn}(\sigma) x_{\sigma(1)} y_{\sigma(2)} z_{\sigma(3)} (x_{\sigma(4)})^{(1)} (y_{\sigma(5)})^{(1)} (z_{\sigma(6)})^{(1)},$$

where σ runs over the group \mathfrak{S}_6 of permutations of $\{1, \dots, 6\}$. Then $f \in \mathcal{O}(V_1)^{G_1}$, but f does not lie in $\mathcal{O}((V//G)_1)$, that is, it cannot be expressed as a polynomial in elements of $\mathcal{O}(V)^G$ and their first derivatives. To see this, note first that f is homogeneous of weight 3 (since each term has three derivatives). Hence there is no contribution to f coming from products of two cubic invariants, since if both cubics were first derivatives, such terms could have weight at most two. For f to lie in $\mathcal{O}((V//G)_1)$, it must be homogeneous of degree three in the quadratics, and each quadratic appearing must be a first derivative of one of the q_{ij} . But this is impossible because q_{ij} is symmetric in i and j , whereas f is totally antisymmetric. Thus V is bad. However, f can be expressed (up to a constant multiple) in the form

$$\sum_{\sigma \in \mathfrak{S}_6} c_{\sigma(1), \sigma(2), \sigma(3)} (c_{\sigma(4), \sigma(5), \sigma(6)})^{(3)},$$

so f lies in $\mathcal{O}((V//G)_3)$. Later (see Theorem 6.3), we will show that $\mathcal{O}(V_\infty)^{G_\infty} = \mathcal{O}((V//G)_\infty)$.

Theorem 3.11. Suppose that $G \subset \mathrm{GL}(V)$ is finite and nontrivial. Then V is m -bad for any $m \geq 1$ and V is not D -finite.

Proof. Note that $G_m = G$ for all m . Let $k > 0$ be minimal such that there is a homogeneous generator of $\mathcal{O}(V)^G$ of degree k . Let f_1, \dots, f_ℓ be a basis of the generators of degree k . Write $V_1 \simeq TV = V \oplus V'$ where $V' \simeq V$ as G -modules. Using the isomorphism we obtain minimal generators f'_1, \dots, f'_ℓ of $\mathcal{O}(V')^G$ which are linearly independent. The f'_i exist in every $\mathcal{O}(V_m)^G$ for $m \geq 1$. They are not in $p_m^* \mathcal{O}((V/G)_m)$ because the only possibility is that $f'_i = D^k f_i$ for all i where the latter have terms involving the variables of V_k not in V_1 . Thus V is m -bad for all $m \geq 1$.

Let f be a homogeneous invariant of $\mathcal{O}(V_\infty)^G$ of minimal positive degree, say m . Let $1 \leq i_1 < \dots < i_m \leq s$. Then there is a polarization f_{i_1, \dots, i_m} which is multilinear and invariant on the copies of V in V_s corresponding to the indices i_1, \dots, i_m . Now consider $i_j = rm + j$ for $r \geq 1$. If $f := f_{i_1, \dots, i_m}$ is in the differential subalgebra of $\mathcal{O}(V_\infty)^G$ generated by $\mathcal{O}(V_{rm})^G$, then we must have that f is a sum of D to some powers applied to invariants of degree m lying in $\mathcal{O}(V_{rm})^G$. But it is easy to see that such a sum can never give f . \square

Corollary 3.12. If X is a smooth affine G -variety and (W, H) is a slice representation of X with H finite and nontrivial, then X is bad and not D -finite.

We need some more background on the action of G , see [Lun]. The points of $V//G$ are in one to one correspondence with the closed orbits Gv , $v \in V$. Let $H := G_v$ be the isotropy group (which is reductive) and let (W, H) be the slice representation. Then the fiber $p^{-1}(p(v))$ is isomorphic to $G \times^H \mathcal{N}(W)$ where $\mathcal{N}(W) := p_W^{-1}(p_W(0))$ is the *null cone* of W . Let (H) denote the conjugacy class of H in G and let $(V//G)_{(H)}$ denote the closed orbits Gv' such that $G_{v'} \in (H)$. Then there are finitely many strata $(V//G)_{(H)}$ each of which is smooth and irreducible. For reductive subgroups H_1 and H_2 of G , write $(H_1) \leq (H_2)$ if H_1 is G -conjugate to a subgroup of H_2 . Then among the isotropy classes of closed orbits, there is a unique minimum (H) with respect to \leq , called the *principal isotropy class*. We also call H a *principal isotropy group* and corresponding closed orbits are called *principal orbits*. Then $(V//G)_{(H)}$ is the unique open stratum in $V//G$ and we also denote it by $(V//G)_{\text{pr}}$. Let Gv be a principal orbit with $G_v = H$. Then the fiber of p through v is of the form $G \times^H W$ where W is the nontrivial part of the slice representation of H at v and $\mathcal{O}(W)^H = \mathbb{C}$. We say that a G -module is *stable* if the general G -orbit is closed. Then the slice representations of the principal isotropy groups are trivial and $V_{\text{pr}} := p^{-1}((V//G)_{\text{pr}})$ is open and consists of principal orbits.

Let S be an irreducible hypersurface in $V//G$. Then the ideal of S is generated by an invariant f . Write $f = f_1^{a_1} \dots f_n^{a_n}$ where the f_i are irreducible polynomials in $\mathcal{O}(V)$. We say that the irreducible component $\{f_i = 0\}$ of $p^{-1}(S)$ is *schematically reduced* if $a_i = 1$. Equivalently, the differential df does not vanish at some point of $\{f_i = 0\}$. We say that $p^{-1}(S)$ is schematically reduced if all of its irreducible components are. Equivalently, f generates the ideal of $p^{-1}(S)$ in $\mathcal{O}(V)$. The *codimension one strata* of V are the inverse images in V of the codimension one strata of $V//G$.

Theorem 3.13. *Let V be a G -module where $\dim V//G = 1$.*

- (1) *If $\mathcal{O}(V_m)^{G_m} = \mathcal{O}(\mathbb{C}_m)$ for some $m \geq 1$, then an irreducible component of $\mathcal{N}(V)$ is schematically reduced.*
- (2) *If an irreducible component of $\mathcal{N}(V)$ is schematically reduced, then $\mathcal{O}(V_m)^{G_m} = \mathcal{O}(\mathbb{C}_m)$ for all $m \geq 1$.*

Proof. Since $\dim V//G = 1$, $V//G \simeq \mathbb{C}$ and $\mathcal{O}(V)^G$ is generated by a homogeneous invariant p . Write $p = p_1^{a_1} \dots p_n^{a_n}$ where the p_i are irreducible polynomials in $\mathcal{O}(V)$. Suppose that no irreducible component of $\mathcal{N}(V)$ is schematically reduced. Then $a_i \geq 2$ for all i and $(Dp)^2$ is divisible by p , yet $(Dp)^2/p$ is not the pull back of an element of $\mathcal{O}(\mathbb{C}_m)$ and we have (1).

Now suppose that an irreducible component of $\mathcal{N}(V)$ is schematically reduced and that V is stable. Let $V' = \{v \in V \mid dp(v) \neq 0\}$. Then V' is G -stable, open and dense in V . Since dp does not vanish somewhere on $\mathcal{N}(V)$, p is a smooth mapping of V' onto \mathbb{C} . Hence $p_m: V'_m \rightarrow \mathbb{C}_m$ is smooth and surjective. The principal fibers of $p: V \rightarrow \mathbb{C}$ are homogeneous spaces G/H where H is reductive. Since $G \rightarrow G/H$ is a principal H -bundle, $G_m \rightarrow (G/H)_m$ is a principal H_m -bundle and $(G/H)_m \simeq G_m/H_m$. It follows that the fibers of p_m in $(V_{\text{pr}})_m$ are homogeneous spaces G_m/H_m . Hence any $h \in \mathcal{O}(V_m)^{G_m}$ is the pull-back of a rational function \tilde{h} on \mathbb{C}_m . If \tilde{h} has poles, then so does $p_m^* \tilde{h} = h$. Hence \tilde{h} is in $\mathcal{O}(\mathbb{C}_m)$ and we have proved (2) in case V is stable.

Now suppose that V is not stable. Then the principal fibers are $G \times^H W$ where $\mathcal{O}(W)^H = \mathbb{C}$. Since $\mathcal{O}(W)^H = \mathbb{C}$, by Hilbert-Mumford there is a 1-parameter subgroup $\lambda: \mathbb{C}^* \rightarrow H$ such that $HW_\lambda = W$ where W_λ is the sum of the strictly positive weight spaces of λ . There is a dense open subset W'_λ of W_λ such that $H \times W'_\lambda \rightarrow W' \subset W$ is surjective and smooth

where W' is open in W . Then $H_m \times W'_{\lambda,m} \rightarrow W'_m$ is surjective and smooth where λ has only positive weights on $W'_{\lambda,m} := (W'_\lambda)_m$. Hence the H_m -invariants of W_m are just the constants. It follows that the G_m -invariants on $G_m \times^{H_m} W_m$ are constants and the proof above goes through. \square

Corollary 3.14 (Eck). *Let V be a stable G -module with $\dim V//G = 1$. Assume that the generating invariant p is irreducible. Then $\mathcal{O}(V_m)^{G_m} = \mathcal{O}(\mathbb{C}_m)$.*

We say that the G -module V is *coregular* if $V//G$ is smooth, equivalently; $\mathcal{O}(V)^G$ is a polynomial ring.

Corollary 3.15. *Let V be coregular. Then $\mathcal{O}(V_m)^{G_m} = \mathcal{O}((V//G)_m)$ if and only if each codimension one stratum of V has a schematically reduced irreducible component.*

Proof. Set $Z := V//G$. If a codimension one stratum has no schematically reduced irreducible component, then the corresponding slice representation is of the form $(W + \theta, H)$ where θ is a trivial representation, $W^H = 0$, $\dim W//H = 1$ and $\mathcal{N}(W)$ has no schematically reduced irreducible component. Then Corollary 3.6 and Theorem 3.13 show that $\mathcal{O}(V_m)^{G_m} \neq \mathcal{O}(Z_m)$.

Now assume that each codimension one stratum has a schematically reduced irreducible component. Let V' be the set of points of V where dp has maximal rank and let $Z' \subset Z$ be the image. Then the complement of Z' has codimension at least 2 in $Z \simeq \mathbb{C}^k$. As in the case $k = 1$, any G_m -invariant polynomial on V_m is the pull-back of a rational function on $(\mathbb{C}^k)_m$ which has no poles on Z'_m . But the complement of Z'_m in $(\mathbb{C}^k)_m$ has codimension at least 2. Hence our G_m -invariant polynomial is the pullback of a polynomial on $(\mathbb{C}^k)_m$. \square

Now let G be a connected complex reductive group and let V be a G -module. We impose a mild technical condition which is automatic if G is semisimple; we assume that $\mathcal{O}(V)$ contains no nontrivial one-dimensional invariant subspaces. Equivalently, we assume that every semi-invariant of G is invariant. The following is well-known.

Lemma 3.16. *Assume that every semi-invariant of $\mathcal{O}(V)$ is invariant.*

- (1) *A function $f \in \mathcal{O}(V)^G$ is irreducible in $\mathcal{O}(V)$ if and only if it is irreducible in $\mathcal{O}(V)^G$. In particular, $\mathcal{O}(V)^G$ is a UFD.*
- (2) *The codimension one strata of V are irreducible and schematically reduced.*
- (3) *Let $S \subset V//G$ have codimension at least 2. Let f_1 and f_2 be relatively prime elements of $\mathcal{O}(V)^G$ which vanish on S . Then f_1 and f_2 are relatively prime elements of $\mathcal{O}(V)$, hence $p^{-1}(S)$ has codimension at least 2 in V .*

Corollary 3.17. *Suppose that V is coregular. Then $\mathcal{O}(V_m)^{G_m} = \mathcal{O}((V//G)_m)$.*

As above, let V' be the set of points in V where dp has maximal rank, and let $Z' \subset Z := V//G$ be the image of V' . Our first goal is to show that $\mathcal{O}(V_m)^{G_m} = \mathcal{O}(Z'_m)$ for all $m \geq 1$.

Lemma 3.18. *Assume that every semi-invariant of $\mathcal{O}(V)$ is invariant. Let S be a codimension one stratum of Z .*

- (1) *The rank of dp is $\dim Z$ on an open dense subset of $p^{-1}(S)$.*
- (2) *$V \setminus V'$ has codimension at least 2 in V .*

Proof. Let $F = p^{-1}(p(v))$ where Gv is closed and $p(v)$ lies in S . Since $\mathcal{O}(V)^G$ is a UFD, the closure of S is defined by an irreducible invariant f . Hence $p^{-1}(\overline{S})$ is irreducible and $df \neq 0$ on an open dense subset of $p^{-1}(S)$. Now F is isomorphic to $G \times^H \mathcal{N}(W)$ where W is the slice representation of $H = G_v$. This fiber is the same everywhere over S and $p^{-1}(S)$ is a fiber bundle over S with fiber F . Thus $f^{-1}(0)$ is schematically reduced if and only if F is schematically reduced, i.e., the G -invariant polynomials vanishing at $p(v)$ generate the ideal of F in $\mathcal{O}(V)$. Thus at a smooth point of F the rank of dp must be maximal. It follows that dp has maximal rank on an open dense subset of $p^{-1}(S)$ and we have (1). By Lemma 3.16(3), if T is a stratum of Z where $\text{codim}_Z T \geq 2$, then $\text{codim}_V p^{-1}(T) \geq 2$. Hence (2) follows from (1). \square

Corollary 3.19. *Let (U, K) be a slice representation of V and let $S = (U \parallel K)_{(H)}$ be a codimension one stratum where $H \subset K$. Then $p_U^{-1}(\overline{S})$ is schematically reduced.*

Proof. Over a point of S , the schematic fiber of p_U is $K \times^H \mathcal{N}(W)$ where W is the slice representation of H . The schematic fiber of p_V is $G \times^H \mathcal{N}(W)$ over points of $S' := (V \parallel G)_{(H)}$. Thus the ranks of dp_U and dp_V are the same on the inverse images of S and S' , respectively, and it follows that the hypersurface $p_U^{-1}(\overline{S})$ is schematically reduced. \square

Proposition 3.20. *Let H be reductive and W an H -module such that the codimension one strata are schematically reduced. Set $Y := W \parallel H$. Suppose that $W' \cap \mathcal{N}(W) \neq \emptyset$. Then W is coregular, $\mathcal{O}(W_m)^{H_m} = \mathcal{O}(Y_m)$ and $p_W(W') = Y$.*

Proof. Since dp_W has maximal rank at a point of $\mathcal{N}(W)$, the image point $0 \in W \parallel H$ is smooth. But Y has a cone structure (induced by the scalar action of \mathbb{C}^* on W). It follows that $W \parallel H$ is smooth, i.e., W is a coregular representation of H . By Corollary 3.15 we have that $\mathcal{O}(W_m)^{H_m} = \mathcal{O}(Y_m)$. Since $W' \cap \mathcal{N}(W) \neq \emptyset$, $p_W(W')$ contains a neighborhood of $0 \in Y$. Since W' and Y are cones, $p_W(W') = Y$. \square

Theorem 3.21. *For all $m \geq 1$, we have $\mathcal{O}(V_m)^{G_m} = \mathcal{O}(Z'_m)$.*

Proof. Let Gv be a closed orbit such that $p^{-1}(p(v))$ intersects V' . Let (W, H) be the slice representation at v . Then Proposition 3.20 and Corollary 3.19 show that $\mathcal{O}(W_m)^{H_m} = \mathcal{O}((W \parallel H)_m)$. Using the slice theorem we see that this implies that $\mathcal{O}(U_m)^{G_m} = \mathcal{O}(\tilde{Z}_m)$ where U is a G -saturated neighborhood of Gv and $\tilde{Z} := p(U)$ is a neighborhood of $p(v)$. Thus given $f \in \mathcal{O}(V'_m)^{G_m}$ there is a unique $h \in \mathcal{O}(Z'_m)$ such that $p_m^* h = f$ on V'_m . Since $\text{codim}_V V \setminus V' \geq 2$, f extends to an element of $\mathcal{O}(V_m)^{G_m}$. \square

4. THE CASE $m = \infty$

In general, it is not true that $\mathcal{O}(Z_m) = \mathcal{O}(Z'_m)$ where $Z := V \parallel G$. However, we will give a criterion for $\mathcal{O}(Z_m) = \mathcal{O}(Z'_m)$ to hold for $m = \infty$. We assume for the rest of this section that G is connected and that V is a G -module such that every semi-invariant is invariant. We also make the following key assumption: $p_\infty: V_\infty \rightarrow Z_\infty$ is surjective.

Proposition 4.1. *Let $f \in \mathcal{O}(V)^G$ such that f is irreducible. Then f generates a primary ideal in $\mathcal{O}(Z_\infty)$.*

Proof. Let $(f)_m$ denote the ideal of f in $\mathcal{O}(Z_m)$ (m finite or ∞) and let (f) denote the ideal of f in $\mathcal{O}(V_\infty)$. It is clear that (f) is prime. Consider the morphism $\mathcal{O}(Z_\infty)/(f)_\infty \rightarrow$

$\mathcal{O}(V_\infty)^{G_\infty}/(f)$. The right hand side is a domain. Let $h \in \mathcal{O}(Z_\infty)$ generate an element in the kernel. Since p_∞ is surjective, h vanishes at the points of Z_∞ where f vanishes. Now h is an element of $\mathcal{O}(Z_m)$ for some m . By Corollary 5.9 of [EM] there is a $p \geq m$ such that $\pi_{p,m}(Z_p) = \psi_m(Z_\infty)$. Thus, considering f and h as functions on Z_p , the zeroes of h contain the zeroes of f . Thus there is a power of h which belongs to $(f)_p$ and $(f)_\infty$ is primary. \square

Corollary 4.2. *We have that $\mathcal{O}(V_\infty)^{G_\infty} = \mathcal{O}(Z'_\infty) = \mathcal{O}(Z_\infty)$.*

Proof. Let f and f' in $\mathcal{O}(Z)$ be relatively prime such that their zero sets contain $Z \setminus Z'$ and have intersection of codimension two (Lemma 3.16(3)). Let $\mathcal{V}(f)$ denote the zero set of f in Z_∞ . Then on $(Z_\infty \setminus \mathcal{V}(f)) \subset Z'_\infty$ we may write $h \in \mathcal{O}(Z'_\infty)$ in the form g/f where $g \in \mathcal{O}(Z_\infty)$. Similarly, we have that $h = g'/f'$. Then gf' lies in $(f)_\infty$. Let f_1 be an irreducible factor of f . Then $(f_1)_\infty$ is primary and no power of f' lies in $(f_1)_\infty$, hence we must have that $g \in (f_1)_\infty$. Dividing out by f_1 and continuing inductively we find that $g \in (f)_\infty$, hence $h \in \mathcal{O}(Z_\infty)$. \square

In the next three sections, we will show that our criterion is satisfied when $G = \mathrm{SL}_n$, GL_n , SO_n , or Sp_{2n} and V is the sum of arbitrarily many copies of the standard representation (and its dual for SL_n and GL_n).

5. THE CASES $G = \mathrm{SL}_n$ AND $G = \mathrm{GL}_n$

First consider the case of the invariant theory of SL_n . We have the representation $V = k(\mathbb{C}^n)^* \oplus \ell \mathbb{C}^n$ where $k, \ell \geq 0$. We may think of the first factor M as all $k \times n$ matrices and the second factor N as all $n \times \ell$ matrices. The invariants are the determinants of the rows of M and of the columns of N and the products of the rows of M with the columns of N . Thus the quotient sits inside $Q \times R \times S$ where Q is all $k \times \ell$ matrices and $R = \wedge^n(\mathbb{C}^k)$ and $S = \wedge^n(\mathbb{C}^\ell)$. We have a group action of GL_k on M on the left and of GL_ℓ on N on the right. The group $H = \mathrm{GL}_k \times \mathrm{GL}_\ell$ acts in the natural way on Q , R and S such that the quotient mapping $p: V \rightarrow Z \subset Q \times R \times S$ is H -equivariant. Let $I = \{i_1 < \dots < i_n\}$ and $J := \{j_1 < \dots < j_n\}$ where $1 \leq i_1 < \dots < i_n \leq k$ and $1 \leq j_1 < \dots < j_n \leq \ell$. Let d_I^* be the function on $R = \wedge^n(\mathbb{C}^k)$ which picks out the coefficient of $e_{i_1} \wedge \dots \wedge e_{i_n}$ and let d_J denote the function on $S = \wedge^n(\mathbb{C}^\ell)$ which picks out the coefficient of $e_{j_1} \wedge \dots \wedge e_{j_n}$. The e_i and e_j denote the usual basis vectors of \mathbb{C}^k and \mathbb{C}^ℓ .

Proposition 5.1. *Let $z = (q, r, s) \in Z \subset (Q \times R \times S)$. Then the following hold ([We]) :*

- (1) *The vectors r and s are either zero or of rank n (hence decomposable).*
- (2) *The matrix q has rank at most n .*
- (3) *For any I and J the corresponding $n \times n$ determinant $d_{I,J}(q)$ of the entries of q equals $d_I^*(r)d_J(s)$.*
- (4) *Consider the natural mappings $\rho_1: Q \otimes R \rightarrow \wedge^{n+1}(\mathbb{C}^k) \otimes \mathbb{C}^\ell$ and $\rho_2: Q \otimes S \rightarrow \wedge^{n+1}(\mathbb{C}^\ell) \otimes \mathbb{C}^k$. Then $\rho_1(Z) = \rho_2(Z) = 0$.*

Let \mathbb{T} denote $\mathrm{Spec} \mathbb{C}[[t]]$.

Lemma 5.2. *Let $z(t) = (q(t), r(t), s(t)) \in Z(\mathbb{T})$.*

- (1) *If $r(t) \neq 0$, then acting by an element of $\mathrm{GL}_k(\mathbb{T})$ we can arrange that $r(t) = t^a e_1 \wedge \dots \wedge e_n$ where $a \geq 0$. It then follows that $q(t)_{i,j} = 0$ for $i > n$.*

- (2) If $s(t) \neq 0$, then acting by an element of $\mathrm{GL}_\ell(\mathbb{T})$ we can arrange that $s(t) = t^b e_1 \wedge \cdots \wedge e_n$ where $b \geq 0$. It then follows that $q(t)_{i,j} = 0$ for $j > n$.

Proof. Suppose that $r(t) \neq 0$. For any element of $\wedge^{n-1}((\mathbb{C}^k)^*)$ we have its contraction with $r(t)$ which gives an element of $\mathbb{C}^k(\mathbb{T})$. The elements obtained in this way form a submodule W of $\mathbb{C}^k(\mathbb{T})$ of rank n , which is free since $\mathbb{C}[[t]]$ is a PID. By a change of basis using $\mathrm{GL}_k(\mathbb{T})$ we can arrange that W is spanned by $t^{m_1} e_1, \dots, t^{m_n} e_n$. Hence our element $r(t)$ is $t^{\sum m_i} e_1 \wedge \cdots \wedge e_n$. Now we can apply the relations 5.1(4) which, in our case, say that the products $r(t)q_{i,j}(t)$ vanish for $i > n$. This gives (1) and the proof of (2) is similar. \square

Theorem 5.3. Let $(V, G) = (k(\mathbb{C}^n)^* \oplus \ell\mathbb{C}^n, \mathrm{SL}_n)$. Then $p_\infty: V_\infty \rightarrow Z_\infty$ is surjective, hence $\mathcal{O}(V_\infty)^{G_\infty} = \mathcal{O}(Z'_\infty) = \mathcal{O}(Z_\infty)$.

Proof. Let $(q(t), r(t), s(t))$ be our element in $Z(\mathbb{T})$.

Case 1: The rank of $q(t)$ is n . Then $r(t)$ and $s(t)$ are not zero by 5.1(3) and by Lemma 5.2 we can assume that $r(t) = t^a e_1 \wedge \cdots \wedge e_n$, $s(t) = t^b e_1 \wedge \cdots \wedge e_n$ and $q_{i,j}(t) = 0$ for i or $j > n$. Moreover, q has rank n . Consider $H_0 := \mathrm{SL}_n(\mathbb{T}) \times \mathrm{SL}_n(\mathbb{T}) \subset \mathrm{GL}_k \times \mathrm{GL}_\ell$ where the copies of SL_n acts on the first n coordinates. Then $H_0(\mathbb{T})$ fixes $r(t)$ and $s(t)$ and under its action we can bring the upper left corner of $q(t)$ to the diagonal form $\mathrm{diag}(c(t)t^{a_1}, t^{a_2}, \dots, t^{a_n})$ where $c(t)$ is a unit. Then by 5.1(3) we must have that $c(t) = 1$ and $\sum a_i = a + b$. Now let the nonzero entries in $m(t)$ be $\mathrm{diag}(t^{a_1-b}, t^{a_2}, \dots, t^{a_n})$ and let the nonzero entries of $n(t)$ be $\mathrm{diag}(t^b, 1, \dots, 1)$ where there are $n - 1$ ones. Then $p_\infty(m(t), n(t)) = z(t)$.

Case 2: The rank of $q(t)$ is less than n . If $r(t)$ and $s(t)$ are zero, then use $H(\mathbb{T})$ to bring $q(t)$ to the form $\mathrm{diag}(t^{a_1}, \dots, t^{a_d}, 0, \dots, 0)$ where $d < n$. Then $(q(t), 0, 0)$ is the image of $(m(t), n(t))$ where $m(t) = q(t)$ in the $d \times d$ upper left hand corner and $n(t)$ is the $d \times d$ identity matrix in the upper left hand corner, and all other entries of $m(t)$ and $n(t)$ are zero.

We are left with the case where, say, $r(t) \neq 0$. Then 5.1(3) implies that $s(t) = 0$. By Lemma 5.2 we may assume that $r(t) = t^a e_1 \wedge \cdots \wedge e_n$ and that $q_{i,j}(t) = 0$ for $i > n$. Acting by $\mathrm{SL}_n(\mathbb{T}) \times \mathrm{GL}_\ell(\mathbb{T})$ we may bring the nonzero entries of $q(t)$ to the form $\mathrm{diag}(t^{a_1}, \dots, t^{a_d})$ where $d < n$. Let the nonzero entries of $m(t)$ be $\mathrm{diag}(1, \dots, 1, t^a)$ where there are $n - 1$ ones and let the nonzero entries of $n(t)$ be $q(t)$ in the $d \times d$ upper left hand corner. Then $(m(t), n(t))$ maps to $(q(t), r(t), 0)$. \square

Now consider the case of the invariant theory of GL_n . We have $V = k(\mathbb{C}^n)^* \oplus \ell\mathbb{C}^n = M \oplus N$ as before. Then the quotient by GL_n is the mapping $p: V \rightarrow Z \subset Q \simeq \mathbb{C}^k \otimes \mathbb{C}^\ell$ where Z is the set of elements of Q of rank at most n . Then as above one shows

Theorem 5.4. Let $(V, G) = (k(\mathbb{C}^n)^* \oplus \ell\mathbb{C}^n, \mathrm{GL}_n)$. Then $p_\infty: V_\infty \rightarrow Z_\infty$ is surjective, hence $\mathcal{O}(V_\infty)^{G_\infty} = \mathcal{O}(Z'_\infty) = \mathcal{O}(Z_\infty)$.

6. THE CASE $G = \mathrm{SO}_n$

We have that $(V, G) = (k\mathbb{C}^n, \mathrm{SO}_n)$. We consider V as the space of $k \times n$ matrices M and the invariants are the inner products of the rows and the determinants of any choice of n rows. The inner product invariants correspond to the map M to S where S is the space of symmetric $k \times k$ matrices and the mapping sends M to MM^T . The determinant invariants land in $Q := \wedge^n(\mathbb{C}^k)$. Note that GL_k acts naturally on M , S and Q and that

$p: V \rightarrow Z \subset S \times Q$ is H -equivariant. Let I be a set of indices $\{1 \leq i_1 < \dots < i_n \leq k\}$ as before.

Proposition 6.1. *Let $z = (s, q) \in Z \subset (S \times Q)$. Then the following hold ([We]) :*

- (1) *The matrix s has rank at most n .*
- (2) *The vector q is zero or of rank n .*
- (3) *For any I , the corresponding symmetric determinant of s is the square of the corresponding function $d_I(q)$.*
- (4) *The mapping $S \otimes Q \simeq S^2(\mathbb{C}^k) \otimes \wedge^n(\mathbb{C}^k) \rightarrow \wedge^{n+1}(\mathbb{C}^k) \otimes \mathbb{C}^k$ restricted to Z is zero.*

Lemma 6.2. *Let $s(t)$ be an $n \times n$ symmetric matrix with entries in $\mathbb{C}[[t]]$. Then there is an $h(t) \in \mathrm{SL}_n(\mathbb{T})$ such that $h(t)s(t)h(t)^T$ is diagonal.*

Proof. For $f(t) \in \mathbb{C}[[t]]$, $f \neq 0$, let the order of f denote the highest power of t dividing f . Let $s_{ij}(t)$ be the ij entry of s and suppose that the first column of $s(t)$ is not zero. Let $h_1(t)$ be a matrix obtained from the identity matrix by replacing the first row by the first row plus a linear combination of all the other rows. Then $h_1(t) \in \mathrm{SL}_n(\mathbb{T})$. Let $s'(t) = h_1(t)s(t)h_1(t)^T$. We may choose $h_1(t)$ such that $s'_{11}(t) \neq 0$ and such that $s'_{11}(t)$ has the least order of the elements in the first column of s' . Thus $s'_{1j}(t)$ is a multiple $a_{1j}(t)$ of $s'_{11}(t)$ for $j > 1$. Let $h_2(t)$ be the matrix obtained from the identity matrix by replacing row j by row j minus $a_{1j}(t)$ times row one for all $j > 1$. Then $h_2(t) \in \mathrm{SL}_n(\mathbb{T})$ and $s''(t) = h_2(t)s'(t)h_2(t)^T$ is of block diagonal form $\begin{pmatrix} a(t) & 0 \\ 0 & r(t) \end{pmatrix}$ where $a(t) \in \mathbb{C}[[t]]$ and $r(t)$ is $(n-1) \times (n-1)$ symmetric. If the first column of $s(t)$ had been zero, then we are automatically in this case with $a(t) = 0$. Now the desired matrix $h(t)$ exists by induction on n . \square

Theorem 6.3. *Let $(V, G) = (k\mathbb{C}^n, \mathrm{SO}_n)$. Then $p_\infty: V_\infty \rightarrow Z_\infty$ is surjective, hence $\mathcal{O}(V_\infty)^{G_\infty} = \mathcal{O}(Z'_\infty) = \mathcal{O}(Z_\infty)$.*

Proof. Let $z(t) = (s(t), q(t)) \in Z(\mathbb{T})$. First suppose that $q(t) \neq 0$. Then as in Lemma 5.2 we may assume that $q(t) = t^a e_1 \wedge \dots \wedge e_n$ and then the relation 6.1(4) shows that $s_{ij}(t)$ vanishes for i or $j > n$ and 6.1(3) shows that $s(t)$ has rank n . Now $q(t)$ is fixed by $\mathrm{SL}_n(\mathbb{T}) \subset \mathrm{GL}_k(\mathbb{T})$ and using $\mathrm{SL}_n(\mathbb{T})$ we may bring $s(t)$ to diagonal form $\mathrm{diag}(c(t)t^{a_1}, t^{a_2}, \dots, t^{a_n})$ where $c(t)$ is a unit. Then since $\det(s(t)) = t^{2a}$ we must have that $c(t) = 1$ and $\sum a_i = 2a$. We may assume that $a_i = 2b_i + 1$ is odd for $1 \leq i \leq d$ and that $a_i = 2b_i$ is even for $i > d$ where d has to be even. Let

$$x(t) = \frac{1}{2} \begin{pmatrix} 1+t & \sqrt{-1}(1-t) \\ -\sqrt{-1}(1-t) & 1+t \end{pmatrix}.$$

Then $\det(x(t)) = t$ and $x(t)x(t)^T = \mathrm{diag}(t, t)$. Now we can form a block diagonal $n \times n$ matrix $y(t)$ using copies of $x(t)$ and ones such that $\det y(t) = t^{d/2}$ and $y(t)y(t)^T = \mathrm{diag}(t, \dots, t, 1, \dots, 1)$ where t occurs d times. Finally, let

$$m(t) = \mathrm{diag}(t^{b_1}, \dots, t^{b_n})y(t).$$

Then $m(t)m(t)^T = s(t)$ and $\det(m(t)) = t^{\sum a_i/2} = q(t)$. We leave the case where $q(t) = 0$ to the reader. \square

7. THE CASE $G = \mathrm{Sp}_{2n}$

We have that $V = k\mathbb{C}^{2n}$. We consider V as the space of $k \times 2n$ matrices M and the invariants are the symplectic products of the rows of M . To be more specific, let J_{2n} be

the $2n \times 2n$ block diagonal matrix

$$J_{2n} = \text{diag}(\kappa, \kappa, \dots, \kappa), \quad \kappa = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We have a mapping $p: M \rightarrow Z \subset R := \wedge^2(\mathbb{C}^k)$ which sends a matrix A to AJA^T . Since $J^T = -J$, the image of p consists of skew symmetric $k \times k$ matrices, which we can identify with R . The image Z of p consists of the elements of R of rank at most $2n$. We have an action of $H := \text{GL}_k$ on the rows of M and the usual action of GL_k on $R = \wedge^2(\mathbb{C}^k)$. The quotient mapping p is H -equivariant.

Theorem 7.1. *Let $(V, G) = (k\mathbb{C}^{2n}, \text{Sp}_{2n})$. Then $p_\infty: V_\infty \rightarrow Z_\infty$ is surjective, hence $\mathcal{O}(V_\infty)^{G_\infty} = \mathcal{O}(Z'_\infty) = \mathcal{O}(Z_\infty)$.*

Proof. Let $r(t) \in Z(\mathbb{T})$. Then acting by $H(\mathbb{T})$ we can bring $r(t)$ to the normal form $c(t^{a_1}e_1 \wedge e_2 + \dots + t^{a_d}e_{2d-1} \wedge e_{2d})$ where $d \leq n$ and $c \neq 0$. Let

$$x_a(t) = \frac{\sqrt{c}}{2} \begin{pmatrix} 1 + t^a & -1 + t^a \\ -1 + t^a & 1 + t^a \end{pmatrix}.$$

Then $x_a(t)\kappa x_a(t)^T = ct^a\kappa$ which we have identified with $ct^ae_1 \wedge e_2$. Thus there is a matrix $m(t)$ with block diagonal entries $x_{a_i}(t)$ such that $p_\infty(m(t)) = r(t)$. \square

8. CONCLUDING REMARKS

The equality $\lim_{m \rightarrow \infty} \mathcal{O}(V_m)^{G_m} = \lim_{m \rightarrow \infty} \mathcal{O}(Z_m)$ is equivalent to V being m -good for $m = \infty$, so we will also say that V is *stably good* in this case. If V is m -good for finite m , it is not true that kV (k copies of V) is m -good for $k \geq 2$, although kV may be stably good. We saw this in the case of SO_3 acting on \mathbb{C}^3 with $k = 6$ (Example 3.10). However, it is usually not true that kV is stably good, even for $k = 2$.

Example 8.1. Let $G = \text{SL}_2$ and $V = R_3$, the space of binary forms of degree 3. Then V is coregular (hence good). The principal isotropy group of V is finite and nontrivial, hence the action of G on $2V$ has a finite isotropy group acting nontrivially on its slice representation. Hence $2V$ is not m -good for any $m \geq 1$.

Remark 8.2. *It is easy to show that the only SL_2 -modules V such that kV is m -good for any $m \geq 1$ for all $k \geq 1$ (or even just for $k = 2$) are of the form ℓR_1 or ℓR_2 (except for trivial factors).*

If V is stably good then V is D -finite. Recall that in Example 3.9 V is conjectured to be D -finite, while it is not stably good. In general, V is not D -finite as we saw for finite G .

REFERENCES

- [E] D. Eck, *Invariants of k -jet actions*, Houston J. Math. Vol. 10, No. 2 (1984) 159-168.
- [EM] L. Ein and M. Mustata, *Jet schemes and singularities*, Algebraic geometry—Seattle 2005. Part 2, 505–546, Proc. Sympos. Pure Math., 80, Part 2, Amer. Math. Soc., Providence, RI, 2009.
- [FBZ] E. Frenkel and D. Ben-Zvi, *Vertex Algebras and Algebraic Curves*, Math. Surveys and Monographs, Vol. 88, American Math. Soc., 2001.
- [Kol] E. Kolchin, *Differential algebra and algebraic groups*, Academic Press, New York 1973.
- [LSS] A. Linshaw, G. Schwarz, and B. Song, *Arc spaces and the vertex algebra commutant problem*, preprint.
- [Lun] D. Luna, *Slices étales*, Sur les groupes algébriques, Soc. Math. France, Paris, 1973, pp. 81–105. Bull. Soc. Math. France, Paris, Mémoire 33.
- [N] J. Nash, *Arc structure of singularities*, Duke Math. J. 81, no. 1, 1995, 31-38.

- [Mu] M. Mustata, *Jet schemes of locally complete intersection canonical singularities*, Invent. Math. 145 (2001), no. 3, 397–424.
- [We] H. Weyl, *The Classical Groups: Their Invariants and Representations*, Princeton University Press, 1939.
- [Zh] Y. Zhu, *Modular invariants of characters of vertex operators*, J. Amer. Soc. 9 (1996) 237-302.

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